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Commuting Self-Adjoint Partial Differential Operators and a Group Theoretic Problem

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In \mathbf{R}^n let Ω denote a Nikodym region (= a connected open set on which every distribution of finite Dirichlet integral is itself in $L^2(\Omega)$). The existence of n commuting self-adjoint operators H_1, \dots, H_n in $L^2(\Omega)$ such that each H_i is a restriction of $-i \partial/\partial x_i$ (acting in the distribution sense) is shown to be equivalent to the existence of a set $A \subset \mathbf{R}^n$ such that the restrictions to Ω of the functions $\exp i \sum \lambda_j x_j$ form a total orthogonal family in $L^2(\Omega)$. If it is required, in addition, that the unitary groups generated by H_1, \dots, H_n act multiplicatively on $L^2(\Omega)$, then this is shown to correspond to the requirement that A can be chosen as a subgroup of the additive group \mathbf{R}^n . The measurable sets $\Omega \subset \mathbf{R}^n$ (of finite Lebesgue measure) for which there exists a subgroup $A \subset \mathbf{R}^n$ as stated are precisely those measurable sets which (after a correction by a null set) form a system of representatives for the quotient of \mathbf{R}^n by some subgroup Γ (essentially the dual of A).

INTRODUCTION

In 1958, Segal posed the following problem to the author: Which are the connected open sets $\Omega \subset \mathbf{R}^n$ such that there exist, on the Hilbert space $L^2(\Omega)$, commuting self-adjoint restrictions H_1, \dots, H_n of the operators $-i \partial/\partial x_1, \dots, -i \partial/\partial x_n$ (acting on $L^2(\Omega)$ in the distribution sense), the commutation being understood in the sense of commuting spectral measures. Segal's motivation was to clarify the relations between formal differential operators given in the region Ω and their interpretations as operators in the Hilbert space $L^2(\Omega)$. While there is a vast literature for the case of an individual operator, there is little in the case of several operators, and the question of the existence of interpretations consistent with the basic algebraic relations between given differential operators seems not to have been treated. When operators H_1, \dots, H_n of the indicated type exist, there is by spectral theory an automatic corresponding interpretation for any

given constant-coefficient linear partial differential operator as a normal operator in $L^2(\Omega)$ and so a particularly satisfactory solution to the interpretation question cited.

The stated problem is affine invariant, and any open cube, hence any open parallelepiped, has the desired property, the self-adjoint restrictions in question being determined, e.g., by suitable periodic boundary conditions.

Restricting the attention to connected open sets $\Omega \subset \mathbf{R}^2$ of finite Lebesgue measure $m(\Omega)$, and satisfying a mild regularity condition (finiteness of the Poincaré constant, cf. Section 3), the author established, as a necessary and sufficient condition on Ω , that there should exist a set $A \subset \mathbf{R}^n$ such that the functions $e_\lambda(x) = \exp(i\lambda x)$, $\lambda \in A$, should form a total (= complete) orthogonal family in $L^2(\Omega)$. In the affirmative case the (simultaneous) spectrum $\sigma(H_1, \dots, H_n)$ serves as such a set A . This result (Theorem I, Section 3) was not published at the time in recognition of the fact that the condition obtained is not a very explicit one. It allowed us, however, to prove that, say in \mathbf{R}^2 , an open circular disc does not have the desired property, nor does a triangle, see Section 4.

Later, Segal has suggested to me that the problem could be modified by adding the natural requirement that each of the self-adjoint operators H_j should generate unitary groups $\exp(i\tau H_j)$, $\tau \in \mathbf{R}$, acting *multiplicatively* on $L^2(\Omega)$ with respect to pointwise multiplication (whenever this does not lead outside $L^2(\Omega)$). For this restricted problem a quite satisfactory solution can be given (Theorem II, Section 7). It turns out that the restriction concerning multiplicativity corresponds to demanding that the exponent set $A = \sigma(H_1, \dots, H_n)$ should be a *subgroup* of \mathbf{R}^n (Lemma 7), necessarily a discrete and free abelian group with n generators. Moreover, a measurable set $\Omega \subset \mathbf{R}^n$ with $0 < m(\Omega) < +\infty$ admits a subgroup A of \mathbf{R}^n such that $(e_\lambda)_{\lambda \in A}$ is a total, orthogonal family in $L^2(\Omega)$ if and only if Ω (after correction on a null set) is a fundamental set for some subgroup Γ of \mathbf{R}^n (again necessarily discrete and with n generators), that is, $\mathbf{R}^n = \Omega + \Gamma$, the sum being direct (Lemma 6). In the affirmative case this latter "translation subgroup" Γ may be taken as $\Gamma = 2\pi A^*$ where A^* denotes the dual of the exponent subgroup $A \subset \mathbf{R}^n$.

Summing up, the only Nikodym regions (= connected open sets of finite Lebesgue measure and finite Poincaré constant) $\Omega \subset \mathbf{R}^n$ which meet the requirements in the modified version of Segal's problem, are those which (after correction by a null set) form a system of representatives for \mathbf{R}^n/Γ for some discrete subgroup $\Gamma \subset \mathbf{R}^n$ with n generators. Examples in the plane ($n = 2$): A square (or a paral-

lellogram), a regular hexagon, or a circular disc less its images under reflection with respect to two adjacent sides of an inscribed square.

Further comments to the original problem are given in Section 8.

It seems plausible that the present results extend to the integrability of given infinitesimal representations of an arbitrary Lie group by vector fields on an open subset of a homogeneous space.

1. GENERAL NOTATIONS

\mathbf{C} , \mathbf{R} , \mathbf{Z} , and \mathbf{N} denote the complex numbers, the reals, the integers, and the natural numbers, respectively. For a fixed dimension $n \in \mathbf{N}$ we denote by m the Lebesgue measure on \mathbf{R}^n . For any (Lebesgue) measurable subset Ω of \mathbf{R}^n with $0 < m(\Omega) < +\infty$ we use

$$(f | g) = \frac{1}{m(\Omega)} \int_{\Omega} f \bar{g} \, dm$$

as a scalar product on the complex Hilbert space $L^2(\Omega)$, and we write $\|f\|^2 = (f | f)$ for $f \in L^2(\Omega)$. For

$$\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbf{R}^n \quad \text{and} \quad x = (x_1, \dots, x_n) \in \mathbf{R}^n$$

we put

$$e_{\lambda}(x) = \exp(i\lambda x), \quad \lambda x = \sum_{j=1}^n \lambda_j x_j.$$

The functions e_{λ} , $\lambda \in \mathbf{R}^n$, will often be considered as defined for $x \in \Omega$ only, in which case they may be considered as elements of $L^2(\Omega)$ because $0 < m(\Omega) < +\infty$. The differential operators

$$D_j = -i\partial/\partial x_j \quad (j = 1, \dots, n)$$

are understood in general to act in the distribution sense on $L^2(\Omega)$. Thus D_j is the "maximal" operator on $L^2(\Omega)$ with the domain

$$\text{dom } D_j = \{u \in L^2(\Omega) \mid D_j u \in L^2(\Omega)\}.$$

The indicator function for a set A is denoted by 1_A .

2. COMMUTING FAMILY OF SELF-ADJOINT OPERATORS

On a complex Hilbert space \mathcal{H} let

$$H = (H_1, \dots, H_n)$$

denote a family of n self-adjoint (not necessarily bounded) operators which commute with one another (in the sense of commuting spectral measures). The domain of this family is denoted by $\text{dom } H$; it is defined as the intersection of the domains of the individual operators:

$$\text{dom } H = \bigcap_{j=1}^n \text{dom } H_j.$$

Denoting by $\xi = (\xi_1, \dots, \xi_n)$ the generic point of \mathbf{R}^n , or the identical mapping of \mathbf{R}^n onto itself, we have the canonical spectral representation

$$H = \int \xi \, dE,$$

that is,

$$H_j = \int \xi_j \, dE \quad (j = 1, \dots, n),$$

of such a family H . Here E denotes the spectral measure on \mathbf{R}^n associated with H . The support of E is the (simultaneous) *spectrum* of H and will be denoted by $\sigma(H)$. It is a closed subset of \mathbf{R}^n . The atomic part of E determines the *point spectrum* $\sigma_p(H)$ of H . It consists of all points $\lambda \in \mathbf{R}^n$ such that $E(\{\lambda\}) \neq 0$, in other words of all (simultaneous) *eigenvalues* for H , an eigenvalue $\lambda = (\lambda_1, \dots, \lambda_n)$ for $H = (H_1, \dots, H_n)$ being a point of \mathbf{R}^n such that the subspace (eigenspace)

$$\mathcal{E}(\{\lambda\}) = \{u \in \text{dom } H \mid Hu = \lambda u\}$$

is $\neq \{0\}$. (The relation $Hu = \lambda u$ means of course $H_j u = \lambda_j u$ for every $j = 1, \dots, n$.) Note that $E(\{\lambda\})$ is the orthogonal projection operator of \mathcal{H} onto $\mathcal{E}(\{\lambda\})$.

The family H is said to have a *pure point spectrum* if the spectral measure E is purely atomic, that is carried by $\sigma_p(H)$ in the sense that $E(\sigma_p(H)) = I$. This amounts to saying that the eigenspaces $\mathcal{E}(\{\lambda\})$, $\lambda \in \sigma_p(H)$, span \mathcal{H} topologically in the sense that their union is total in \mathcal{H} , in other words that their Hilbert sum is all of \mathcal{H} . It follows then that $\sigma(H)$ is the closure of $\sigma_p(H)$.

A sufficient (but not necessary) condition for H to have a pure point spectrum is that the spectrum $\sigma(H)$ is discrete (or just countable) as a subset of \mathbf{R}^n . We call a set $A \subset \mathbf{R}^n$ *discrete* if every point of A is isolated in A .

With the above notations we have the following lemma.

LEMMA. *Let P denote a finite dimensional orthogonal projection*

operator on \mathcal{H} , and let $\lambda = (\lambda_1, \dots, \lambda_n)$ denote a given point of the spectrum $\sigma(H)$. Suppose that there exists a finite constant C such that

$$\|u - Pu\|^2 \leq C \sum_{j=1}^n \|H_j u - \lambda_j u\|^2 \quad (1)$$

for all $u \in \text{dom } H (= \bigcap_{j=1}^n \text{dom } H_j)$. Then

$$0 \neq E(\{\lambda\}) \leq P,$$

that is, λ is an eigenvalue for H with eigenspace $\mathcal{E}(\{\lambda\})$ contained in the range of P . If $E(\{\lambda\}) = P$ then λ is an isolated point of $\sigma(H)$, the distance between λ and $\sigma(H) \setminus \{\lambda\}$ being $\geq 1/C^{1/2}$.

Proof. For $p = 1, 2, \dots$ write

$$\delta_p = \{\xi \in \mathbf{R}^n \mid |\xi - \lambda| < 1/p\}.$$

From $\lambda \in \sigma(H)$ follows $E(\delta_p) \neq 0$. Choose $u_p \in E(\delta_p)\mathcal{H}$ so that $\|u_p\| = 1$, and note that $u_p \in \text{dom } H$, and

$$\sum_{j=1}^n \|H_j u_p - \lambda_j u_p\|^2 = \int_{\delta_p} |\xi - \lambda|^2 d\|Eu_p\|^2 \leq 1/p^2.$$

It follows that, for any $j = 1, \dots, n$

$$H_j u_p - \lambda_j u_p \rightarrow 0 \text{ strongly in } \mathcal{H} \quad \text{as } p \rightarrow \infty, \quad (2)$$

and further by virtue of (1) that

$$u_p - Pu_p \rightarrow 0 \text{ strongly in } \mathcal{H}. \quad (3)$$

Passing, if necessary, to a suitable subsequence, we may assume that there is a vector $u \in \mathcal{H}$ such that

$$u_p \rightarrow u \text{ weakly in } \mathcal{H}, \quad (4)$$

and hence

$$Pu_p \rightarrow Pu \text{ strongly in } \mathcal{H} \quad (5)$$

because $P\mathcal{H}$ is finite dimensional. Combining (3) and (5) we infer that $u_p \rightarrow Pu$ strongly, hence weakly, and so $Pu = u$ in view of (4). Consequently $u_p \rightarrow u$ strongly, from which it follows by (2) that $u \in \text{dom } H_j$ and $H_j u = \lambda_j u$ for each $j = 1, \dots, n$ because H_j is a closed operator. This shows that $E(\{\lambda\}) \neq 0$ since $\|u\| = \lim \|u_p\| = 1$. For any $u \in E(\{\lambda\})\mathcal{H}$ we have $H_j u = \lambda_j u$ for every $j = 1, \dots, n$, and hence it follows from (1) that $u - Pu = 0$, that is $u \in P\mathcal{H}$.

Suppose now, in addition to (1), that $E(\{\lambda\}) = P$, and write

$$\delta = \{\xi \in \mathbf{R}^n \mid 0 < |\xi - \lambda| < \rho\}$$

for some fixed ρ such that $0 < \rho < 1/C^{1/2}$. For any $u \in E(\delta)\mathcal{H}$ we obtain $u \in \text{dom } H$ and

$$\sum_{j=1}^n \|H_j u - \lambda_j u\|^2 = \int_{\delta} |\xi - \lambda|^2 d\|Eu\|^2 \leq \rho^2 \|u\|^2.$$

On the other hand,

$$\|u - Pu\|^2 = \|u\|^2,$$

since $Pu = E(\{\lambda\})u = 0$ because $\lambda \notin \delta$. It follows now from (1) that $u = 0$ because $\rho^2 < 1/C$. Consequently $E(\delta) = 0$, and so $\delta \cap \sigma(H) = \emptyset$, from which it follows that the distance between λ and $\sigma(H) \setminus \{\lambda\}$ is $\geq 1/C^{1/2}$. ■

3. NIKODYM REGIONS

From Deny-Lions [2, p. 328 ff.] we adopt the following definition.

DEFINITION. A (nonvoid) open subset Ω of \mathbf{R}^n is called a Nikodym set if every distribution u on Ω such that all $D_j u$ are in $L^2(\Omega)$ ($j = 1, \dots, n$), is itself in $L^2(\Omega)$.

A connected Nikodym set is called a Nikodym region.

Note that a Nikodym set has necessarily finite measure (take $u = 1$). Using the scalar product and norm on $L^2(\Omega)$ as normalized in Section 1, and further the notation e_λ introduced there (in particular $e_0 =$ the constant 1 on Ω), we have the following result essentially contained in [2].

LEMMA. *A connected open set $\Omega \subset \mathbf{R}^n$ is a Nikodym region if and only if $m(\Omega) < +\infty$ and there is a finite constant $C = C(\Omega)$ such that the following two equivalent conditions are fulfilled with u ranging over the Sobolev space $\{u \in L^2(\Omega) \mid D_1 u, \dots, D_n u \in L^2(\Omega)\}$.¹*

¹ It suffices, however, to let u range over $\mathcal{C}^\infty(\Omega) \cap \bigcap_{i=1}^\infty \text{dom } D_i$, since it is known that the first order system $D = (D_1, \dots, D_n)$, acting in the distribution sense on $L^2(\Omega)$, is the closure of its restriction to the above space of smooth functions u . (For this, see e.g. Fuglede [4, Chapter II, Section 2] using the idea of the proof of a similar result in Deny-Lions [2, p. 312].)

$$(i) \quad \|u\|^2 \leq |(u | e_0)|^2 + C(\Omega) \sum_{j=1}^n \|D_j u\|^2,$$

$$(ii) \quad \|u - (u | e_\lambda) e_\lambda\|^2 \leq C(\Omega) \sum_{j=1}^n \|D_j u - \lambda_j u\|^2$$

for some, and hence for any $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbf{R}^n$.

Proof. The fact that (i) characterizes the Nikodym regions is due to Deny-Lions [2, Theorem 5.3]. Of course (i) may be given the equivalent form

$$\|u - (u | e_0) e_0\|^2 \leq C(\Omega) \sum_{j=1}^n \|D_j u\|^2$$

because $u \mapsto (u | e_0) e_0$ defines an orthogonal projection operator on the Hilbert space $L^2(\Omega)$. Replacing, for given $\lambda \in \mathbf{R}^n$, u by $u/e_\lambda = u\bar{e}_\lambda = ue_{-\lambda}$ in this inequality, we obtain (ii). In the opposite direction replace u by ue_λ . ■

Remarks. (1) The inequality (i) is the Poincaré inequality, and the smallest possible value of the constant $C(\Omega)$ is called the *Poincaré constant* for Ω (cf.) Courant-Hilbert [1, pp. 488, 511] and Deny-Lions [2, p. 329]). Under suitable regularity conditions on Ω the Poincaré constant is the smallest positive eigenvalue for $-\Delta$ (Δ = the Laplacian) with the homogeneous Neumann condition of vanishing normal derivative (cf. [2, p. 340 ff.]).

(2) Any bounded open set which is starshaped with respect to one of its points, is a Nikodym region [2, p. 332].

(3) Any finite union of Nikodym sets is a Nikodym set. More generally, an open set $\Omega \subset \mathbf{R}^n$ is a Nikodym set if Ω differs only by a null set (for Lebesgue measure on \mathbf{R}^n) from some finite union of Nikodym sets contained in Ω . (This is an immediate consequence of the above definition of a Nikodym set.)

THEOREM I. *Let $\Omega \subset \mathbf{R}^n$ be a Nikodym region.*

(a) *Let $H = (H_1, \dots, H_n)$ denote a commuting family (if any) of self-adjoint restrictions H_j of D_j on $L^2(\Omega)$, $j = 1, \dots, n$. Then H has a discrete spectrum, each point $\lambda \in \sigma(H)$ being a simple eigenvalue for H with the eigenspace $\mathbf{C}e_\lambda$, and hence $(e_\lambda)_{\lambda \in \sigma(H)}$ is an orthonormal base for $L^2(\Omega)$. Moreover, $\sigma(H) = \sigma_p(H) = \{\lambda \in \mathbf{R}^n \mid e_\lambda \in \text{dom } H\}$.*

(b) *Conversely, let Λ denote a subset (if any) of \mathbf{R}^n such that $(e_\lambda)_{\lambda \in \Lambda}$ is an orthonormal base for $L^2(\Omega)$. Then there exists a unique commuting family $H = (H_1, \dots, H_n)$ of self-adjoint restrictions H_j of D_j on $L^2(\Omega)$ with the property that $\{e_\lambda \mid \lambda \in \Lambda\} \subset \text{dom } H$, or equivalently that $\Lambda = \sigma(H)$.²*

² For the uniqueness it suffices even to assume $\Lambda \subset \sigma(H)$.

Proof. (a) Consider first, on $L^2(\Omega)$, any commuting family $H = (H_1, \dots, H_n)$ of self-adjoint restrictions of D_1, \dots, D_n , and denote by E the associated spectral measure on \mathbf{R}^n (Section 2). Then it follows from the above lemma that Lemma 2 applies to $\mathcal{H} = L^2(\Omega)$, taking for P the one-dimensional projection P_λ of $L^2(\Omega)$ on $\mathbf{C}e_\lambda$ for any $\lambda \in \sigma(H)$, whereby $P_\lambda u = (u | e_\lambda)e_\lambda$ for every $u \in L^2(\Omega)$. It follows that

$$E(\{\lambda\}) = P_\lambda \quad \text{for every } \lambda \in \sigma(H),$$

because the range $\mathbf{C}e_\lambda$ of P_λ is one-dimensional. By the final assertion of Lemma 2, $\sigma(H)$ is discrete, and hence H has a pure point spectrum (Section 2), and $\sigma(H) = \sigma_p(H)$. For any $\lambda \in \mathbf{R}^n$ we now have the (bi)implications

$$[\lambda \in \sigma(H)] \Rightarrow [e_\lambda \in \text{dom } H] \Rightarrow [He_\lambda = \lambda e_\lambda] \Rightarrow [\lambda \in \sigma(H)], \quad (6)$$

showing that

$$\sigma(H) = \{\lambda \in \mathbf{R}^n \mid e_\lambda \in \text{dom } H\} = \{\lambda \in \mathbf{R}^n \mid He_\lambda = \lambda e_\lambda\}.$$

(b) Next let Λ denote a subset (if any) of \mathbf{R}^n such that $(e_\lambda)_{\lambda \in \Lambda}$ is an orthonormal base for $L^2(\Omega)$. It follows from (6) that $\Lambda \subset \sigma(H)$ holds (with H as above) if and only if $\{e_\lambda \mid \lambda \in \Lambda\} \subset \text{dom } H$. In the affirmative case we must have $\Lambda = \sigma(H)$ because $\{e_\lambda \mid \lambda \in \sigma(H)\}$ is an orthonormal extension of $\{e_\lambda \mid \lambda \in \Lambda\}$.

As to the uniqueness of $H = (H_1, \dots, H_n)$, each H_j must satisfy $H_j e_\lambda = \lambda_j e_\lambda$ for every $\lambda \in \mathbf{R}^n$ such that $e_\lambda \in \text{dom } H$ because H_j should be the restriction of the maximal operator D_j on $L^2(\Omega)$. In particular, $H_j e_\lambda = \lambda_j e_\lambda$ should hold for every $\lambda \in \Lambda$ when we want that $\{e_\lambda \mid \lambda \in \Lambda\} \subset \text{dom } H$. Since $\{e_\lambda \mid \lambda \in \Lambda\}$ is supposed to be an orthonormal base for $L^2(\Omega)$, H_j must be the closure of its restriction H_j^0 to the subspace M of $L^2(\Omega)$ spanned algebraically by $\{e_\lambda \mid \lambda \in \Lambda\}$, and this restriction H_j^0 is of course uniquely determined from Λ .

Finally, as to the existence of H , we define accordingly each H_j as the closure of the restriction H_j^0 of D_j to the above subspace M . Then H_j is self-adjoint because $H_j e_\lambda = \lambda_j e_\lambda$ for every $\lambda \in \Lambda$. Moreover, H_j is a restriction of the maximal operator D_j since D_j is closed. Clearly $H = (H_1, \dots, H_n)$ is a commuting family, and

$$\{e_\lambda \mid \lambda \in \Lambda\} \subset M \subset \text{dom } H.$$

Remarks. (1) The hypothesis that Ω be a Nikodym region is not needed in Part (b) of the above theorem. It suffices for that to assume that Ω is open and that $0 < m(\Omega) < +\infty$.

On the other hand, part (a) breaks down if we allow Ω to be disconnected. This follows, e.g., from the following example in the one-dimensional case (easily extendable to higher dimensions). Take for Ω the union of the open intervals $(0, 2)$ and $(3, 4)$. Then $(e_\lambda | e_\mu) = 0$ means

$$z^2 - 1 + z^4 - z^3 = 0$$

for $z = e^{i(\lambda - \mu)} (\neq 1)$. But this reduces to $z^3 + z + 1 = 0$, an equation with no roots of modulus $|z| = 1$. Hence $L^2(\Omega)$ contains no two orthogonal functions of the form e_λ . Nevertheless, there does exist a self-adjoint restriction H_1 of D_1 on $L^2(\Omega)$ because this is the case for each component of Ω (see also the next remark).

(2) The class of all open sets $\Omega \subset \mathbf{R}^n$ with $0 < m(\Omega) < +\infty$ such that there exists on $L^2(\Omega)$ a commuting family of self-adjoint restrictions of D_j ($j = 1, \dots, n$) is stable under *disjoint union*. (We omit the easy proof.) Note, however, that a union Ω of disjoint open sets Ω_p may belong to the class in question without the individual Ω_p being of that class. Example in the plane \mathbf{R}^2 : Let Ω_1 be an open triangle or disc, and let $\Omega_2 = S \setminus \bar{\Omega}_1$ with S an open square in \mathbf{R}^2 containing the closure $\bar{\Omega}_1$ of Ω_1 . Then Ω_1 and Ω_2 are Nikodym regions, and Ω_1 is *not* in the stated class (see Section 4), but $\Omega_1 \cup \Omega_2$ is in this class because this union only differs by a null set from S (which is in the class), and hence $L^2(\Omega_1 \cup \Omega_2) = L^2(S)$. A less trivial example is obtained by replacing Ω_2 by a suitable translate thereof and applying Theorem II.

(3) Let Ω denote any measurable subset of \mathbf{R}^n such that $0 < m(\Omega) < +\infty$. Any set $\mathcal{A} \subset \mathbf{R}^n$ such that $(e_\lambda)_{\lambda \in \mathcal{A}}$ is an orthonormal base for $L^2(\Omega)$ must be *discrete* and *closed* in \mathbf{R}^n and cannot be contained in any proper affine subspace of \mathbf{R}^n . In verifying this we may assume that $0 \in \mathcal{A}$, the hypothesis on \mathcal{A} being translation invariant. In view of Lebesgue's bounded convergence theorem $(e_\lambda | e_0)$ is continuous in $\lambda \in \mathbf{R}^n$ and equals 1 at the origin hence differs from 0 for all sufficiently small values of $|\lambda|$. Clearly this implies that \mathcal{A} is discrete and closed, the distance between any two distinct points of \mathcal{A} being bounded from below by some positive constant. If \mathcal{A} were contained in some proper, say linear subspace of \mathbf{R}^n , there would exist a unit vector $a \in \mathbf{R}^n$ such that $a\lambda = 0$ for every $\lambda \in \mathcal{A}$. Then any element of $L^2(\Omega)$ would be representable by a function constant along every line parallel to a because this applies to each member of the orthonormal base $(e_\lambda)_{\lambda \in \mathcal{A}}$. This conclusion, however, is impossible (take, e.g., $f(x) = ax/|x|$, considered for $x \in \Omega$).

4. EXAMPLES

We give two examples of a Nikodym region Ω in the plane \mathbf{R}^2 such that there is, for $L^2(\Omega)$, no pair (H_1, H_2) of commuting self-adjoint restrictions of (D_1, D_2) , or equivalently such that $L^2(\Omega)$ contains no total orthonormal family of the form $(e_\lambda)_{\lambda \in A}$ with $A \subset \mathbf{R}^2$.

(1) *A triangle.* By the affine invariance of the problem we may take

$$\Omega = \{(x, y) \in \mathbf{R}^2 \mid x > 0, y > 0, x + y < 1\}.$$

We show that although there do exist infinite orthonormal families of the form $(e_\lambda)_{\lambda \in A}$ on $L^2(\Omega)$, none of these families is total.

For $\lambda = (\alpha, \beta) \in \mathbf{R}^2$, simple calculations show that the relation

$$(e_\lambda \mid e_0) = 2 \int e^{i(\alpha x + \beta y)} dx dy = 0 \quad (7)$$

is equivalent to the following set of four conditions: $\alpha \neq 0$, $\beta \neq 0$, $\alpha \neq \beta$, and

$$(\alpha - \beta) - \alpha e^{i\beta} + \beta e^{i\alpha} = 0. \quad (8)$$

Since a triangle with the sides $|\alpha - \beta|$, $|\alpha e^{i\beta}| = |\alpha|$, and $|\beta e^{i\alpha}| = |\beta|$ must be degenerate, we obtain $e^{i\alpha} = 1$ or -1 , and similarly for $e^{i\beta}$. Of these four possibilities only $e^{i\alpha} = e^{i\beta} = 1$ conforms with (8). The solutions of (7) are therefore given by

$$\lambda = 2\pi(p, q) \quad \text{with } p, q \in \mathbf{Z} \setminus \{0\} \text{ and } p \neq q. \quad (9)$$

Hence $(e_\lambda)_{\lambda \in A}$ is orthonormal in $L^2(\Omega)$ if and only if every $\lambda \in A - A$ with $\lambda \neq 0$ fulfills (9). In particular, the projection

$$A \ni (\alpha, \beta) \mapsto \alpha \in \mathbf{R} \quad (10)$$

is injective.

As an example, the following set A leads to an infinite orthonormal family $(e_\lambda)_{\lambda \in A}$ in $L^2(\Omega)$:

$$A = \{(2p\pi, -2p\pi) \mid p \in \mathbf{Z}\}.$$

Finally we consider an arbitrary set $A \subset \mathbf{R}^2$ such that $(e_\lambda)_{\lambda \in A}$ is an orthonormal family in $L^2(\Omega)$, and we prove that this family is not total in $L^2(\Omega)$. Proceeding by contradiction we apply Parseval's formula

$$\|f\|^2 = \sum_{\lambda \in A} |(f \mid e_\lambda)|^2 \quad (11)$$

to $f = ge_{\lambda_0}$ for given $\lambda_0 = 2\pi(p_0, q_0)$ with $p_0, q_0 \in \mathbf{Z}$. Here g denotes the indicator function for the square (contained in Ω) given by $0 < x < \frac{1}{2}, 0 < y < \frac{1}{2}$. Clearly g and f may be viewed as elements of $L^2(\Omega)$, and $\|f\|^2 = \frac{1}{2}$, using the normalization from Section 1. Evaluating explicitly $|(f|e_\lambda)|$ for $\lambda = 2\pi(p, q)$ with $p, q \in \mathbf{Z}$ it is easily shown that at least one of the following five points must belong to $(2\pi)^{-1}\mathcal{A}$:

$$(p_0, q_0), (p_0 + 1, q_0), (p_0 - 1, q_0), (p_0, q_0 + 1), (p_0, q_0 - 1),$$

since otherwise

$$\sum_{\lambda \in \mathcal{A}} |(f|e_\lambda)|^2 \leq \frac{3}{4} - \frac{4}{\pi^2} < \frac{1}{2} = \|f\|^2$$

after some computation using the formula

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}.$$

To complete the proof, choose five consecutive integers q_1, \dots, q_5 so that none of the points $(0, 2\pi q_j)$ belongs to \mathcal{A} . (This is possible by the injectivity of the map (10).) By the above result applied to $\lambda_0 = (0, 2\pi q_j)$ for $j = 2, 3$, and 4 , we find that, for each of these 3 values of j , one of the points $(1, q_j)$ or $(-1, q_j)$ belongs to $(2\pi)^{-1}\mathcal{A}$. Clearly this contradicts the injectivity of the map (10).

(2) *A circle.* For the unit disc Ω in \mathbf{R}^2 , every orthonormal family on $L^2(\Omega)$ of the form $(e_\lambda)_{\lambda \in \mathcal{A}}$ must be finite. We show here that \mathcal{A} can contain three elements, but probably no more.

Using polar coordinates r, θ in \mathbf{R}^2 , we find for any two distinct vectors $\lambda, \mu \in \mathbf{R}^2$, writing $|\lambda - \mu| = \rho$,

$$(e_\lambda | e_\mu) = \frac{2}{\rho^2} \int_0^\rho J_0(z) z \, dz = \frac{2}{\rho} J_1(\rho),$$

where J_0 and J_1 are the Bessel functions of order 0 and 1 (cf. Watson [5, Sections 2.1, 2.2]). Thus $(e_\lambda)_{\lambda \in \mathcal{A}}$ is orthonormal in $L^2(\Omega)$ if and only if the distance between any two distinct points of \mathcal{A} is a zero of J_1 . Hence three points of \mathbf{R}^2 forming an equilateral triangle whose side is a positive zero of J_1 will lead to an orthonormal family $(e_\lambda)_{\lambda \in \mathcal{A}}$. On the other hand, a set \mathcal{A} of four points in \mathbf{R}^2 probably cannot fulfill the stated condition because the six distances between distinct points of \mathcal{A} cannot all be equal, and there is a nontrivial algebraic

relation between them. This would lead to an algebraic relation between some of the positive zeros of J_1 , and that is highly unlikely.

Without entering into this difficult question, one can, however, prove that there does not exist any *infinite* set $A \subset \mathbf{R}^2$ such that the distance between any two distinct points of A is a zero of J_1 . We shall not bring the details of the proof, which is based on the asymptotic form of the zeros ρ_n of J_1 (see [5, Section 15.53]):

$$\rho_n = \pi \left(n + \frac{1}{4} + O\left(\frac{1}{n}\right) \right).$$

5. DISCRETE SUBGROUPS OF \mathbf{R}^n

It is well known that a subgroup Γ of the additive group \mathbf{R}^n is discrete if and only if Γ is isomorphic to \mathbf{Z}^k for some $k \in \mathbf{N} \cup \{0\}$. In the affirmative case Γ is a closed subset of \mathbf{R}^n , and k is the dimension of the linear subspace of \mathbf{R}^n spanned by Γ . We shall mainly consider the case $k = n$ where Γ is *total* in \mathbf{R}^n . Explicitly, there exist generators $a^{(1)}, \dots, a^{(n)} \in \mathbf{R}^n$ such that the mapping

$$\mathbf{Z}^n \ni p = (p_1, \dots, p_n) \mapsto \sum_{j=1}^n p_j a^{(j)} \in \mathbf{R}^n$$

is a group isomorphism of \mathbf{Z}^n onto Γ .

To any total discrete subgroup Γ of \mathbf{R}^n corresponds a unique *dual* group $\Lambda = \Gamma^*$, likewise total and discrete, defined as the set of all vectors $\lambda \in \mathbf{R}^n$ such that

$$\lambda \gamma \in \mathbf{Z} \quad \text{for every } \gamma \in \Gamma$$

(where $\lambda \gamma = \sum_{j=1}^n \lambda_j \gamma_j$). Moreover, Γ is the dual of Λ in the same sense, $\Gamma^{**} = \Gamma$.

For any set of generators $a^{(1)}, \dots, a^{(n)}$ for Γ (as above) we obtain a set of generators $b^{(1)}, \dots, b^{(n)}$ for the dual group Λ by taking e.g. the *biorthogonal complement* of $a^{(1)}, \dots, a^{(n)}$, that is, the uniquely determined set of vectors $b^{(1)}, \dots, b^{(n)}$ in \mathbf{R}^n satisfying

$$a^{(j)} b^{(k)} = \delta_{jk} \quad (j, k = 1, \dots, n).$$

By a *fundamental set* for a discrete subgroup Γ of \mathbf{R}^n (acting as a group of translations of \mathbf{R}^n) is understood a set $\Omega \subset \mathbf{R}^n$ such that

$$\Omega + \Gamma = \mathbf{R}^n \quad (\text{direct sum}),$$

that is, a set Ω of representatives for the cosets in $\mathbf{R}^n \pmod{\Gamma}$, one for each coset.

Every discrete subgroup Γ of \mathbf{R}^n has a fundamental set Ω which is measurable with respect to Lebesgue measure m on \mathbf{R}^n . Clearly $m(\Omega)$ is >0 and independent of the choice of Ω . Moreover, $m(\Omega) < +\infty$ if and only if Γ is total.³ In that case the parallelepiped

$$\left\{ \sum_{j=1}^n t_j a^{(j)} \mid t = (t_1, \dots, t_n) \in [0, 1)^n \right\}$$

is a bounded, measurable fundamental set for Γ for any choice of $a^{(1)}, \dots, a^{(n)}$ as a set of generators for Γ .

When Γ is a total, discrete subgroup of \mathbf{R}^n we denote by $L^2(\mathbf{R}^n/\Gamma)$ the vector space of all (equivalence classes of) locally square integrable (with respect to m) functions $f: \mathbf{R}^n \rightarrow \mathbf{C}$ such that, for every $\gamma \in \Gamma$,

$$f(x + \gamma) = f(x)$$

for (almost) every $x \in \mathbf{R}^n$. For any measurable fundamental set Ω for Γ

$$(f \mid g) = \frac{1}{m(\Omega)} \int_{\Omega} f \bar{g} \, dm, \quad f, g \in L^2(\mathbf{R}^n/\Gamma),$$

defines a scalar product on $L^2(\mathbf{R}^n/\Gamma)$ which is independent of the choice of Ω . The restriction mapping $f \mapsto f|_{\Omega}$ is then an isometric isomorphism of $L^2(\mathbf{R}^n/\Gamma)$ onto $L^2(\Omega)$ with the scalar product defined in Section 1. In particular, $L^2(\mathbf{R}^n/\Gamma)$ is a Hilbert space.

LEMMA. *Let Γ denote a total, discrete subgroup of \mathbf{R}^n . Then $\{e_{\lambda} \mid \lambda \in 2\pi\Gamma^*\}$ is an orthonormal base for $L^2(\mathbf{R}^n/\Gamma)$. For any measurable fundamental set $\Omega \subset \mathbf{R}^n$ for Γ the restrictions $e_{\lambda}|_{\Omega}$, $\lambda \in 2\pi\Gamma^*$, form an orthonormal base for $L^2(\Omega)$.*

Proof. The latter assertion follows from the former because the restriction mapping $f \mapsto f|_{\Omega}$ is an isometric isomorphism of $L^2(\mathbf{R}^n/\Gamma)$ onto $L^2(\Omega)$. To establish the former assertion note that $e_{\lambda} \in L^2(\mathbf{R}^n/\Gamma)$ holds if (and only if) $\lambda\gamma \in 2\pi\mathbf{Z}$ for every $\gamma \in \Gamma$, that is if $\lambda \in 2\pi\Gamma^*$. Let

³ If Γ has $k < n$ generators we may assume that the n th coordinate γ_n equals 0 for every $\gamma \in \Gamma$. For almost every $\xi \in \mathbf{R}$ the projection Ω_{ξ} of the intersection of Ω with the hyperplane $x_n = \xi$ onto the (x_1, \dots, x_{n-1}) -space is then a measurable fundamental set with respect to Γ (considered as a subgroup of the (x_1, \dots, x_{n-1}) -space. The $(n-1)$ -dimensional Lebesgue measure of Ω_{ξ} is hence constant for almost every ξ which is impossible by Fubini's theorem because $0 < m(\Omega) < +\infty$.

$a^{(1)}, \dots, a^{(n)}$ denote a set of generators for Γ , and let $b^{(1)}, \dots, b^{(n)}$ form the biorthogonal complement thereof. The linear mapping

$$\mathbf{R}^n \ni t = (t_1, \dots, t_n) \rightarrow \sum_{j=1}^n t_j a^{(j)} \in \mathbf{R}^n$$

induces an isometric isomorphism of $L^2(\mathbf{R}^n/\Gamma)$ onto $L^2(\mathbf{R}^n/\mathbf{Z}^n)$ carrying $\{e_\lambda \mid \lambda \in 2\pi\Gamma^*\}$ onto $\{e_{2\pi p} \mid p \in \mathbf{Z}^n\}$ in accordance with the relation $\lambda = 2\pi \sum_{j=1}^n p_j b^{(j)}$. The assertion of the lemma has thus been reduced to the well-known case $\Gamma = \mathbf{Z}^n$, where $\Gamma^* = \mathbf{Z}^n$. ■

6. CHARACTERIZATIONS OF THE FUNDAMENTAL SETS

LEMMA. *Let Ω denote a measurable subset of \mathbf{R}^n with $0 < m(\Omega) < +\infty$, let Γ denote a total discrete subgroup of \mathbf{R}^n , and write $\Lambda = 2\pi\Gamma^*$, where Γ^* denotes the dual of Γ . The following statements are equivalent:*

- (i) $\{e_\lambda \mid \lambda \in \Lambda\}$ is an orthonormal base for $L^2(\Omega)$.
- (ii) The restriction mapping $f \mapsto f|_\Omega$ is an isometry of $L^2(\mathbf{R}^n/\Gamma)$ onto $L^2(\Omega)$.
- (iii) $\sum_{\gamma \in \Gamma} 1_{\Omega+\gamma} = 1$ almost everywhere in \mathbf{R}^n .
- (iv) Ω is equivalent to a fundamental set Ω' for Γ .

Proof. (i) \Rightarrow (ii). According to Lemma 5, $\{e_\lambda \mid \lambda \in \Lambda\}$ is an orthonormal base for $L^2(\mathbf{R}^n/\Gamma)$. Let φ denote the unique isometric isomorphism of $L^2(\mathbf{R}^n/\Gamma)$ onto $L^2(\Omega)$ for which $\varphi(e_\lambda) = e_\lambda$ (more precisely $\varphi(e_\lambda) = e_\lambda|_\Omega$) for every $\lambda \in \Lambda$. For any $f \in L^2(\mathbf{R}^n/\Gamma)$ we have, writing $c_\lambda = (f|e_\lambda)$ for $\lambda \in \Lambda$,

$$\begin{aligned} f &= \sum_{\lambda \in \Lambda} c_\lambda e_\lambda \quad \text{in } L^2(\mathbf{R}^n/\Gamma), \\ \varphi(f) &= \sum_{\lambda \in \Lambda} c_\lambda e_\lambda \quad \text{in } L^2(\Omega). \end{aligned} \tag{12}$$

Since convergence of a sequence in $L^2(\mathbf{R}^n/\Gamma)$ implies pointwise convergence almost everywhere for a suitable subsequence, there exists a decomposition $\Lambda = \bigcup_{q \in \mathbf{N}} \Lambda_q$ of Λ into disjoint, finite sets Λ_q such that

$$\sum_{q \in \mathbf{N}} \left(\sum_{\lambda \in \Lambda_q} c_\lambda e_\lambda \right) = f \quad \text{pointwise a.e. in } \mathbf{R}^n,$$

in particular pointwise a.e. in Ω . On the other hand, it follows from (12) that

$$\sum_{q \in \mathbf{N}} \left(\sum_{\lambda \in \mathcal{A}_q} c_\lambda e_\lambda \right) = \varphi(f) \quad \text{in } L^2(\Omega),$$

and we conclude that $\varphi(f) = f$ almost everywhere in Ω , that is, $\varphi(f) = f|_\Omega$ in $L^2(\Omega)$.

(ii) \Rightarrow (iii). Writing $E = \mathbf{R}^n \setminus (\Omega + \Gamma)$ we have $1_E \in L^2(\mathbf{R}^n/\Gamma)$ and $E \cap \Omega = \emptyset$. It follows (with $\varphi(f) = f|_\Omega$) that

$$\varphi(1_E) = 1_E|_\Omega = 1_{E \cap \Omega}|_\Omega = 0,$$

and hence by (ii) $1_E = 0$ (in $L^2(\mathbf{R}^n/\Gamma)$), that is $m(E) = 0$. This shows $\sum_{\gamma \in \Gamma} 1_{\Omega + \gamma} \geq 1$ a.e. in \mathbf{R}^n . It remains to prove that the set $E_\gamma = \Omega \cap (\Omega + \gamma)$ is a null set for every $\gamma \in \Gamma \setminus \{0\}$.⁴ For the subset $F = E_\gamma \setminus (E_\gamma - \gamma)$ of Ω we have $1_F \in L^2(\Omega)$, and hence there exists by (ii) $f \in L^2(\mathbf{R}^n/\Gamma)$ such that $f|_\Omega = 1_F$ (a.e.) in Ω . Since

$$(F - \gamma) \subset (E_\gamma - \gamma) = [(\Omega - \gamma) \cap \Omega] \subset (\Omega \setminus F),$$

it follows that $f(x - \gamma) = 0$ (a.e.) in F . On the other hand, $f(x - \gamma) = f(x)$ for (a.e.) $x \in \mathbf{R}^n$, in particular $f(x - \gamma) = 1$ for (a.e.) $x \in F$, and consequently $m(F) = 0$. Since

$$m(E_\gamma - \gamma) = m(E_\gamma) \leq m(\Omega) < +\infty,$$

we infer that the sets E_γ and $E_\gamma - \gamma$ are equivalent. Let now S denote any measurable fundamental set (e.g. a parallel strip) for the subgroup \mathbb{Z}_γ of \mathbf{R}^n :

$$S + \mathbb{Z}_\gamma = \mathbf{R}^n \quad (\text{direct sum}).$$

The measurable sets $E_\gamma \cap (S + p\gamma)$, $p \in \mathbf{Z}$, are then disjoint and cover E_γ . They all have the same measure $m(E_\gamma \cap S)$ because E_γ is equivalent to $E_\gamma + p\gamma$, and hence $E_\gamma \cap (S + p\gamma)$ to $(E_\gamma + p\gamma) \cap (S + p\gamma) = (E_\gamma \cap S) + p\gamma$. Since $m(E_\gamma) \leq m(\Omega) < +\infty$ we conclude that $m(E_\gamma \cap (S + p\gamma)) = 0$ for every $p \in \mathbf{Z}$, and consequently $m(E_\gamma) = 0$.

⁴ For then $(\Omega + \alpha) \cap (\Omega + \beta) = ((\Omega + \alpha - \beta) \cap \Omega) + \beta$ is a nullset for every $(\alpha, \beta) \in \Gamma \times \Gamma$ with $\alpha \neq \beta$, and hence so is

$$\left\{ x \in \mathbf{R}^n \mid \sum_{\gamma \in \Gamma} 1_{\Omega + \gamma}(x) > 1 \right\} = \bigcup_{(\alpha, \beta) \in \Gamma \times \Gamma, \alpha \neq \beta} (\Omega + \alpha) \cap (\Omega + \beta).$$

(iii) \Rightarrow (iv). First replace Ω by the equivalent set

$$\Omega_0 := \Omega \setminus \bigcup_{\gamma \in \Gamma \setminus \{0\}} [\Omega \cap (\Omega + \gamma)]$$

to get a direct sum $\Omega_0 + \Gamma = \mathbf{R}^n \setminus N$. Clearly, $m(N) = 0$, and $N + \Gamma = N$. For any measurable fundamental set A for Γ the sum $(N \cap A) + \Gamma$ is direct, and equal to N since, for any $\gamma \in \Gamma$,

$$(N \cap A) + \gamma = (N + \gamma) \cap (A + \gamma) = N \cap (A + \gamma).$$

The set $\Omega' := \Omega_0 \cup (N \cap A)$ has the stated properties.

(iv) \Rightarrow (i). We may assume that Ω is itself a measurable fundamental set for Γ , and so Lemma 5 is applicable. ■

7. THE MULTIPLICATIVE CASE

Returning to the situation in Theorem I, Section 3, we consider, for a Nikodym region $\Omega \subset \mathbf{R}^n$, a commuting family (if any) $H = (H_1, \dots, H_n)$ ($= \int \xi dE$) of self-adjoint restrictions H_j of D_j on $L^2(\Omega)$. This family generates an n parameter group of unitary operators

$$U(t) = \prod_{j=1}^n \exp(it_j H_j) \left(= \int e_t dE \right), \quad t = (t_1, \dots, t_n),$$

on $L^2(\Omega)$. We say that this unitary group U acts *multiplicatively* if

$$U(t)(fg) = [U(t)f][U(t)g]$$

for every $t \in \mathbf{R}^n$ and for every $f, g \in L^2(\Omega)$ such that $fg \in L^2(\Omega)$. (Clearly this property reduces to the multiplicativity of each of the one parameter groups $U_j(\tau) = \exp(i\tau H_j)$, $j = 1, \dots, n$, $\tau \in \mathbf{R}$.)

LEMMA. *With the notations of Theorem I (a), the unitary group U generated by H acts multiplicatively on $L^2(\Omega)$ if and only if $\Lambda := \sigma(H)$ is a subgroup of \mathbf{R}^n (necessarily total and discrete).*

Proof. Suppose first that U acts multiplicatively, and let $\lambda, \mu \in \Lambda$. Since $e_{\lambda-\mu}$, e_μ , and $e_{\lambda-\mu}e_\mu = e_\lambda$ all belong to $L^2(\Omega)$, we obtain

$$[U(t)e_{\lambda-\mu}][U(t)e_\mu] = U(t)e_\lambda,$$

showing that

$$U(t)e_{\lambda-\mu} = \exp[i(\lambda - \mu)t]e_\lambda,$$

because $U(t)e_\lambda = \exp(i\lambda t)e_\lambda$, etc. It follows that $e_{\lambda-\mu}$ is an eigenvector for the generator $H = (H_1, \dots, H_n)$ corresponding to the eigenvalue $\lambda - \mu$, and hence $\lambda - \mu \in \Lambda = \sigma(H)$.

Conversely suppose that $\Lambda = \sigma(H)$ is a subgroup of \mathbf{R}^n . Then Λ is discrete and total by Theorem I and the third remark to it because the functions e_λ , $\lambda \in \Lambda$ (considered on Ω) form an orthonormal base for $L^2(\Omega)$. According to Lemma 6, Ω is equivalent to a fundamental set Ω' for $\Gamma := 2\pi\Lambda^*$. Moreover, $(e_\lambda)_{\lambda \in \Lambda}$ is an orthonormal base for $L^2(\mathbf{R}^n/\Gamma)$ by Lemma 5. Let φ denote the isometric isomorphism of $L^2(\mathbf{R}^n/\Gamma)$ onto $L^2(\Omega)$ defined by $\varphi(f) =$ the restriction of f to Ω (or equivalently to Ω'). The n parameter unitary group

$$V(t) = \varphi^{-1}U(t)\varphi, \quad t \in \mathbf{R}^n,$$

acting on $L^2(\mathbf{R}^n/\Gamma)$, has the property

$$V(t)e_\lambda = e^{i\lambda t}e_\lambda, \quad \lambda \in \Lambda,$$

because $U(t)(e_\lambda | \Omega) = e^{i\lambda t}(e_\lambda | \Omega)$. This means that $(V(t)f)(x) = f(x + t)$ for every $f = e_\lambda$, $\lambda \in \Lambda$, and hence for all $f \in L^2(\mathbf{R}^n/\Gamma)$. This implies that $V(t)$ acts multiplicatively on $L^2(\mathbf{R}^n/\Gamma)$, and hence $U(t)$ acts multiplicatively on $L^2(\Omega)$ because the isometric restriction mapping φ is multiplicative in the analogous sense, that is $\varphi(fg) = \varphi(f)\varphi(g)$ for any $f, g \in L^2(\mathbf{R}^n/\Gamma)$ such that $fg \in L^2(\mathbf{R}^n/\Gamma)$. ■

THEOREM II. *The following three statements are equivalent for any Nikodym region $\Omega \subset \mathbf{R}^n$:*

(i) *There exists a commuting family $H = (H_1, \dots, H_n)$ of self-adjoint restrictions of $D = (D_1, \dots, D_n)$ on $L^2(\Omega)$ such that the unitary operators*

$$U(t) := \prod_{j=1}^n \exp(it_j H_j), \quad t = (t_1, \dots, t_n) \in \mathbf{R}^n,$$

act multiplicatively on $L^2(\Omega)$.

(ii) *There exists a (necessarily total and discrete) subgroup Λ of \mathbf{R}^n such that $(e_\lambda)_{\lambda \in \Lambda}$ is an orthonormal base for $L^2(\Omega)$.*

(iii) *There exists a (necessarily total and discrete) subgroup Γ of \mathbf{R}^n such that Ω is equivalent to a fundamental set Ω' for Γ .*

In the affirmative case the relations $\sigma(H) = \Lambda = 2\pi\Gamma^$ define a 1-1 correspondence between the objects H , Λ , and Γ of (i), (ii), and*

(iii), respectively. Moreover, $U(t)f$ is determined for any $t \in \mathbf{R}^n$ and any $f \in L^2(\Omega)$ by

$$(U(t)f)(x) = f(\omega(x + t))$$

for almost every $x \in \Omega$, where $\omega(y)$ for any $y \in \mathbf{R}^n$ denotes the unique representative of $y + \Gamma$ in Ω' . For any $j = 1, \dots, n$ the domain of H_j consists of all restrictions to Ω of functions $u \in L^2(\mathbf{R}^n/\Gamma)$ such that $D_j u \in L^2(\mathbf{R}^n/\Gamma)$ (in the distribution sense).⁵

Proof. The subgroups Λ and Γ in (ii) and (iii), respectively, must be total and discrete (Remark 3, Section 3, and note 3, Section 5). The equivalence of (i) and (ii) follows from Theorem I and Lemma 7; that of (ii) and (iii) from Lemma 6. Suppose now that H exists with the properties stated in (i), and let $U(t)$, $t \in \mathbf{R}^n$, denote the unitary group on $L^2(\Omega)$ generated by H . Further let $\Lambda = \sigma(H)$, and $\Gamma = 2\pi\Lambda^*$. With the notations of the last part of the proof of Lemma 7 above, the corresponding group $V(t) = \varphi^{-1}U(t)\varphi$ acting on $L^2(\mathbf{R}^n/\Gamma)$ is induced by the translations $x \mapsto x + t$, that is,

$$(V(t)g)(x) = g(x + t),$$

for every $g \in L^2(\mathbf{R}^n/\Gamma)$. For any function $f \in L^2(\Omega)$, considered as the restriction $f = \varphi(g)$ of a function $g \in L^2(\mathbf{R}^n/\Gamma)$, we thus obtain for $x \in \Omega$

$$(U(t)f)(x) = (\varphi V(t)g)(x) = g(x + t) = f(\omega(x + t))$$

The one parameter translation group $V_j(\tau) = \varphi^{-1}U_j(\tau)\varphi$, $\tau \in \mathbf{R}$, on $L^2(\mathbf{R}^n/\Gamma)$ has the infinitesimal generator $\varphi^{-1}H_j\varphi$. This generator, however, is known to be the self-adjoint, maximal operator D_j on $L^2(\mathbf{R}^n/\Gamma)$, whence the assertion concerning the domain of H_j . ■

Remark. Remarks 1 and 2 after Theorem I carry over mutatis mutandis to the present multiplicative case. In particular, the hypothesis that Ω be a Nikodym region is not needed for the implication (ii) \Rightarrow (i) in Theorem II (and of course not for the equivalence between (ii) and (iii), as established in Lemma 6).

⁵ More explicitly, $u \in L^2(\mathbf{R}^n/\Gamma)$ should be (equivalent to) a function absolutely continuous on almost every line parallel to the x_j -axis, and furthermore $\partial u / \partial x_j$ should be of class $L^2(\mathbf{R}^n/\Gamma)$ (or, equivalently, its restriction to Ω should be of class $L^2(\Omega)$), cf. Deny-Lions [2, p. 313 ff.].

8. FURTHER REMARKS TO THE ORIGINAL PROBLEM

We shall now return briefly to Segal's original problem, which was reduced in Theorem I, Section 3 (for Nikodym regions), to the question regions), to the question of characterizing those Nikodym regions in \mathbf{R}^n which are "spectral sets" according to the following definition.

DEFINITION. A measurable set $\Omega \subset \mathbf{R}^n$ with $0 < m(\Omega) < +\infty$ is called a *spectral set* if there exists a set $A \subset \mathbf{R}^n$ such that $(e_\lambda)_{\lambda \in A}$ is an orthonormal base for $L^2(\Omega)$. Any such set A is called an *exponent set* for Ω .

DEFINITION. A measurable set $\Omega \subset \mathbf{R}^n$ with $0 < m(\Omega) < +\infty$ is called a *direct summand* if there exists a set $\Gamma \subset \mathbf{R}^n$ such that (after correction of Ω on a null set)

$$\Omega + \Gamma = \mathbf{R}^n \quad (\text{direct sum}).$$

Any such set Γ is called a *translation set* for Ω .

It is easily shown that any such translation set Γ is discrete, closed, and total in \mathbf{R}^n (just like any exponent set for a spectral set).

In \mathbf{R}^2 a triangle is an example of a set which is neither a spectral set (Section 4) nor a direct summand; another such example is a circular disc. According to Lemma 6 a set $\Omega \subset \mathbf{R}^n$ is a spectral set admitting an exponent *group* A if and only if Ω is a direct summand admitting a translation *group* Γ .

Conjecture. A measurable set $\Omega \subset \mathbf{R}^n$ with $0 < m(\Omega) < +\infty$ is a spectral set if and only if Ω is a direct summand.

EXAMPLE. In the case $n = 1$ the union Ω of the intervals $]0, 1[$ and $]2, 3[$ is a spectral set and a direct summand in \mathbf{R} . It is easily verified that, for example,

$$A = \left\{0, \frac{\pi}{2}\right\} + 2\pi\mathbf{Z}, \quad \Gamma = \{0, 1\} + 4\mathbf{Z}$$

serve as an exponent set A and a translation set Γ for Ω . No subgroup of \mathbf{R} is an exponent set or a translation set for Ω .

In higher dimensions a similar example is derived by taking the product set with a single interval for each of the remaining coordinates. For $n \geq 3$ it is even possible to use the same idea to construct a *connected* open set $\Omega \subset \mathbf{R}^n$ (the interior of the union of 12 translates of a cube, hence a Nikodym region) which is a spectral set and a direct

summand of the non-trivial kind where A and Γ cannot be chosen as subgroups. This shows that Segal's original problem is strictly more general than the modified problem in Section 7. (No such example exists for $n = 1$ and presumably not for $n = 2$.)

The notions of a spectral set and a direct summand as defined above have obvious extensions to the case where \mathbf{R}^n is replaced by any locally compact abelian group G . In order to test the likelihood of the above conjecture it is natural to try to find all spectral sets and all direct summands in the case of the group \mathbf{Z} or the finite cyclic groups \mathbf{Z}_n of order $n \in \mathbf{N}$. Both of these enterprises seem to be quite difficult. While some results are known concerning the decompositions of a finite group into a direct sum (or product in the non commutative case) of subsets (cf. Fuchs [4, Chap. XV] and references therein), I have come across no information at all concerning the spectral subsets. By a simple "pigeon hole" argument it is shown in Fuchs [4, p. 317] that any translation set for a (finite) direct summand in \mathbf{Z} is periodic. This reduces the decomposition problem for \mathbf{Z} to that of the finite cyclic groups.

My own attempts have led to the following modest results:

(1) A set $A \subset \mathbf{Z}$ (or in \mathbf{Z}_n) of at most four elements is a spectral set if and only if A is a direct summand.

(2) Every direct summand A in \mathbf{Z} (or in \mathbf{Z}_n), the number of elements of which is a power of a prime, is a spectral set in \mathbf{Z} (resp. in \mathbf{Z}_n).

(3) Every direct summand A in \mathbf{Z}_n is a spectral set for any n such that \mathbf{Z}_n has the following property:

(P) For any decomposition $\mathbf{Z}_n = A_1 + A_2$ of \mathbf{Z}_n into a direct sum, at least one of the sets A_1 or A_2 is *periodic*. (A subset A of \mathbf{Z}_n is called periodic if there exists an element $g \in \mathbf{Z}_n$ of order >1 such that $g + A = A$.)

According to the work of Hajós, Rédei, de Bruijn, and Sands (see Fuchs [4, p. 317]), \mathbf{Z}_n has the property (P) if and only if n is a divisor in some integer of one of the forms $p^s q$, $p^2 q^2$, $p^2 q r$, or $p q r s$, where p , q , r , and s denote distinct primes. Thus \mathbf{Z}_{72} is the cyclic group of least order *not* having the property (P). This is exhibited by the following example in $G = \mathbf{Z} \pmod{72}$, due to Hajós (see Fuchs [4, p. 316]):

$$A_1 = \{0, 8, 16, 24, 32\} \pmod{72},$$

$$A_2 = \{0, 6, 12, 18, 24, 30, 36, 42, 48, 54, 60, 66\} \pmod{72}$$

Then $G = A_1 + A_2$ (direct sum), but neither A_1 nor A_2 is periodic. Nevertheless, A_1 and A_2 are both spectral sets. The following subsets of $(2\pi/72)\mathbf{Z} \pmod{2\pi}$ are exponent sets for A_1 and A_2 , respectively:

$$A_1 = \frac{2\pi}{72} \{0, 6, 12, 18, 24, 30\} \left(\text{or } \frac{2\pi}{72} \{0, 3, 6, 18, 21, 24\} \right) \pmod{2\pi}.$$

$$A_2 = \frac{2\pi}{72} \{0, 4, 8, 9, 13, 17, 36, 40, 44, 45, 49, 53\} \pmod{2\pi}.$$

Thus Hajós' example does not violate our conjecture.

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